Method of Characteristics

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The main technique for solving PDE's in AME 500B is the "*eigenfunction*" method. When the domain is unbounded, the method is also called the "*transform*" method. These methods are applicable only to linear PDE's because the notion of superposition is central.

Eigenfunction methods are especially successful and powerful when the "spatial part" of the operator pertaining to a PDE is self-adjoint. One of the most frequent and important operators in the (simplest) equations of engineering science is $-d^2/dx^2$.

The standard form of a linear ODE is $d\mathbf{y}/dt = A(t)\mathbf{y} + \mathbf{f}(t)$. We solved this system in the special case, $A = \text{const}$. A direct generalization of this equation to PDE's leads to

$$\boxed{\sum_{i=1}^{n} A_i(\mathbf{x})\,\partial\mathbf{u}/\partial x_i = A(\mathbf{x})\mathbf{u} + \mathbf{f}(\mathbf{x})}$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)$ is the independent variable, $\mathbf{u} = \mathbf{u}(\mathbf{x}) = (u_1, u_2, ..., u_d)$ is the dependent variable, $\mathbf{f} = \mathbf{f}(\mathbf{x})$ is the given forcing term and A_i, A are given square (coefficient) matrices whose orders are the same as the number of dependent variables in $\mathbf{u}$. In a typical physical problem, $\mathbf{x} = (t, x, y, z)$ and the components of $\mathbf{u}$ may be temperature, pressure, density, velocity components, etc. The boxed equation above is of great value because many physical equations have this form. The presence of several matrices makes life difficult.

There is also a subtle problem. For illustration consider the 1D wave equation, $u_{tt} = u_{xx}$, and the 2D Laplace equation, $-u_{yy} = u_{xx}$. We can reduce these equations to first order PDE; let $u_t - f = 0$ and $u_x - g = 0$. The wave equation becomes $\varepsilon f_t - g_x = 0$, $\varepsilon = +1$. A similar result holds for the Laplace equation with t replaced by y and $\varepsilon = -1$. We now introduce the dependent variable, $\mathbf{u} = (u, f, g)$, and independent variables $\mathbf{x} = (t, x)$ so that $\mathbf{f} = 0$ and

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \varepsilon & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Now the Laplacian is a self-adjoint operator in $\mathbf{x}-$space so the boxed equation can be solved by an eigenfunction method for $\varepsilon = -1$. This method will **not** work in $\mathbf{x} = (t, x)$ space for $\varepsilon = +1$. In other words, minor changes in the coefficient matrices require different methods of solution (because of different auxiliary conditions). In fact, it is not at all obvious how to specify the auxiliary conditions (i.e., boundary and initial conditions) for the general PDE in order to make it into a well posed problem.

In order to get a feel for the PDE, we replace the matrices by scalars. This is a good beginning for the introduction of the method of characteristics that can be extended to the matrix version of the PDE under certain conditions. What is remarkable is that this method works equally well for nonlinear equations. Also, the method has a nice geometric interpretation in $\mathbf{x}-$ space.

First Order PDE – Characteristics

Consider a first order PDE of the form $a_i \partial u / \partial x_i = b$ where the summation convention over repeated subscripts is understood; the independent variables are $(x_1, x_2, ..., x_n) = \mathbf{x}$ and the dependent variable is $u = u(x_1, x_2, ..., x_n) = u(\mathbf{x})$. It is convenient to think of the independent variables as a vector, $\mathbf{x}$, in $n-$ dimensional space. A similar remark holds for the coefficients, $(a_1, a_2, ..., a_n) = \mathbf{a}$.

Classification of First Order PDE

In the event that $\mathbf{a} = \mathbf{a}(\mathbf{x})$ and $b = b(\mathbf{x})$, the PDE is said to be **linear** because the differential operator on the left-hand side of the PDE is linear. The reader should verify that superposition and proportionality (scaling) holds for the dependent variable, u. The given $\mathbf{a}(\mathbf{x})'s$ are the coefficients of the equation and the given $b(\mathbf{x})$ is the forcing term. When $\mathbf{x} = (x, y)$, an example is $\sin(x+y)u_x + y^2 u_y = \exp(xy) + 2x^3$ with the notation $u_x = \partial u / \partial x$ and $u_y = \partial u / \partial y$.

In the event that some (possibly all) of the $\mathbf{a}'s$ and b also depend on u, the PDE is said to be **quasi-linear**. Many equations of engineering science are quasi-linear (e.g., the inviscid equations of fluid mechanics), though they usually involve several independent variables. When $\mathbf{x} = (x, y)$, an example is $[\sin(x+y) + u^3]u_x + [y^2 + \cosh u]u_y = u^2 \exp(xy) + 2x^3$ with the notation $u_x = \partial u / \partial x$ and $u_y = \partial u / \partial y$.

In the event that some (possibly all) of the $\mathbf{a}'s$ and b also depend on $\mathbf{p} = (\partial u / \partial x_1, \partial u / \partial x_2, ..., \partial u / \partial x_n)$, the PDE is said to be **nonlinear**. In this case, it is generally impossible to write the equation in "coefficient" form. The most general form of first order PDE is simply $G(\mathbf{x}, u, \mathbf{p}) = 0$ for some function G. When $\mathbf{x} = (x, y)$, an example is $\exp(xu + u_x^2) + \sin(yuu_y) + 2(x+y) = 0$.

In order to get our feet wet, we shall consider linear PDE's in one dependent variable. We wish to illustrate the method of characteristics.

Method of Characteristics

The PDE is equivalent to an ODE via the concept of characteristics. There is some nice geometry here. Consider a curve in the underlying $n-$ dimensional vector space whose parametric equation is $\mathbf{x} = \mathbf{x}(\sigma)$, where σ is the parameter. We now set $d\mathbf{x} / d\sigma = \mathbf{a}(\mathbf{x})$; curves defined this way are the **characteristics** of the PDE. The tangent to a characteristic at any point along the curve is the coefficient vector, $\mathbf{a}(\mathbf{x})$.

From calculus we find that the rate of change of u along a characteristic is given by $du/d\sigma = (\partial u/\partial x_i)dx_i/d\sigma = a_i\partial u/\partial x_i = b(\mathbf{x})$ because of the PDE. In other words, along a characteristic, the PDE reduces to the ODE, $du/d\sigma = b(\mathbf{x})$. Hence the PDE is equivalent to the ODE $d\mathbf{x}/d\sigma = \mathbf{a}(\mathbf{x})$ and $du/d\sigma = b(\mathbf{x})$ (and conversely). We immediately see that there is no difficulty in treating the quasi-linear case [for which $\mathbf{a} = \mathbf{a}(\mathbf{x},u)$ and $b = b(\mathbf{x},u)$]. The initial condition for the ODE is stated as $\mathbf{x} = \mathbf{x}_0$ and $u = u_0$ at $\sigma = 0$ (say).

Because of initial conditions for the ODE, we say that the characteristic direction is *time-like* in $\mathbf{x}-$space. At each point, $\mathbf{x}$, this direction is defined by the direction of the coefficient vector $\mathbf{a}(\mathbf{x})$.

We write the formal solution for a characteristic as $\mathbf{x} = \mathbf{x}(\sigma,\mathbf{x}_0)$, whose inverse is $\mathbf{x}_0 = \mathbf{x}_0(\sigma,\mathbf{x})$. The solution for the dependent variable is $u = u_0 + \int_0^\sigma b\left(\mathbf{x}(\sigma',\mathbf{x}_0)\right)d\sigma'$.

Properly Posed Problem for the PDE

A characteristic curve is one dimensional. In order to obtain the solution, $u = u(\mathbf{x})$, in all of $\mathbf{x}-$space by the method described above (called the **method of characteristics**), we must specify $\mathbf{x}_0$ and u_0 on an $(n-1)-$dimensional (hyper-) surface. We call this surface the *initial surface*; it is conveniently specified by an equation of the form, $F(\mathbf{x}) = 0$, that imposes one constraint among the $n-$coordinates of $\mathbf{x} = (x_1, x_2, ..., x_n)$.

We remark that the orientation of the initial surface is not entirely arbitrary; we shall see the explicit constraint in the special example below. Suffice it to state here that the initial surface must be *space-like.*

The properly posed problem for the PDE includes an initial condition, on a space-like surface $F(\mathbf{x}) = 0$, given as $u = g(\mathbf{x})$, where g is an arbitrary function.

We are now ready to put together the pieces. Since by definition $\mathbf{x}_0$ lies on the initial surface, we have the equation $F\left(\mathbf{x}_0(\sigma,\mathbf{x})\right) = 0$. The solution of this equation determines σ for any point in $\mathbf{x}-$space. Roughly, for a given $\mathbf{x}$, the parameter σ indicates how far the point lies from the initial surface – the answer is "the" zero of $F\left(\mathbf{x}_0(\sigma,\mathbf{x})\right) = 0$ which we denote by $\sigma = \sigma(\mathbf{x})$.

Having determined $\sigma(\mathbf{x})$, we now find the intersection of the (unique) characteristic passing through $\mathbf{x}$ and the initial surface. This intersection is "simply" the inverse function, $\mathbf{x}_0 = \mathbf{x}_0\left(\sigma(\mathbf{x}),\mathbf{x}\right)$.

From the last equation of the previous section, the solution of the properly posed problem for the PDE is

$$\boxed{u = u_0 + \int_0^{\sigma} b\left(\mathbf{x}(\sigma', \mathbf{x}_0)\right) d\sigma' = g\left(\mathbf{x}_0\left(\sigma(\mathbf{x}), \mathbf{x}\right)\right) + \int_0^{\sigma(\mathbf{x})} b\left(\mathbf{x}\left(\sigma', \mathbf{x}_0\left(\sigma(\mathbf{x}), \mathbf{x}\right)\right)\right) d\sigma' = u(\mathbf{x})}$$

This is because $u_0 = g(\mathbf{x}_0)$; σ and $\mathbf{x}_0$ are eliminated in favor of $\sigma(\mathbf{x})$ and $\mathbf{x}_0 = \mathbf{x}_0\left(\sigma(\mathbf{x}), \mathbf{x}\right)$, respectively.

Although the equation above is an elegant formal solution, there are some practical difficulties with its use. First, although the PDE is linear, the ODE $d\mathbf{x} / d\sigma = \mathbf{a}(\mathbf{x})$, in general, is nonlinear. Therefore, the characteristics are curves with nonzero curvature and it is usually impossible to obtain closed form analytical solutions for them. Graphical methods (usually employed in simple problems of steady two-dimensional supersonic flows) or numerical methods must be used. Even if the characteristics are straight lines (i.e., the PDE has constant coefficients, $\mathbf{a} = \text{const}$), the determination of the intersection of these lines with the initial surface, $F = 0$, is a nontrivial task because it requires finding the solution of a nonlinear algebraic equation for $\sigma(\mathbf{x})$. Fortunately, the entire problem becomes completely tractable algebraically when the initial surface is a hyper-plane of dimension $(n-1)$. We now work out these details in order to illustrate explicitly the steps of the method of characteristics.

Example: Constant Coefficient PDE and the Initial Surface is a Plane

The well posed problem for the linear PDE: $a_i \partial u / \partial x_i = b(\mathbf{x}), \mathbf{a} = \text{const}$ and $u = g(\mathbf{x})$ on the initial surface, $F(\mathbf{x}) = \boldsymbol{\gamma} \cdot \mathbf{x} - \Gamma = 0$, where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_n) = \text{const}$ and $\Gamma = \text{const}$. Clearly, this initial surface is a plane with surface normal $\boldsymbol{\gamma}$.

The characteristic equation, $d\mathbf{x} / d\sigma = \mathbf{a}$, is trivially integrable; $\mathbf{x} = \sigma\mathbf{a} + \mathbf{x}_0$ whose inverse is equally simple, $\mathbf{x}_0 = \mathbf{x} - \sigma\mathbf{a}$.

Because the initial point, $\mathbf{x}_0$, lies on the initial surface, $F(\mathbf{x}_0) = 0$; we have $\boldsymbol{\gamma} \cdot \mathbf{x} - \sigma(\boldsymbol{\gamma} \cdot \mathbf{a}) - \Gamma = 0$. The solution of the last equation is $\sigma = \sigma(\mathbf{x}) = (\boldsymbol{\gamma} \cdot \mathbf{x} - \Gamma) / (\boldsymbol{\gamma} \cdot \mathbf{a})$ provided that $\boldsymbol{\gamma} \cdot \mathbf{a} \neq 0$. The last inequality ensures that the characteristic (time-like) direction (defined by $\mathbf{a}$) does not lie in the initial surface. In other words, the initial surface does **not** contain the time-like direction; for this reason we call this surface space-like. If the initial surface contains the vector, $\mathbf{a}$, the problem is unsolvable in $\mathbf{x}-$space; the problem is ill-posed.

The solution of the PDE is given by the boxed equation of the last section. This solution reduces to

$$u(\mathbf{x}) = g\left(\mathbf{x} - \sigma(\mathbf{x})\mathbf{a}\right) + \int_0^{\sigma(\mathbf{x})} b\left(\left(\sigma' - \sigma(\mathbf{x})\right)\mathbf{a} + \mathbf{x}\right) d\sigma'$$

where $\sigma(\mathbf{x}) = (\boldsymbol{\gamma} \cdot \mathbf{x} - \Gamma) / (\boldsymbol{\gamma} \cdot \mathbf{a})$. It is a simple matter to verify that $u(\mathbf{x})$ given above satisfies the PDE for any g and b. On the initial surface $\sigma = 0$ so that $u = g(\mathbf{x})$, as required.