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Green's Functions

Consider a domain \mathcal{V} with boundary S whose unit outward normal is \mathbf{n} ; we use the notation $\partial/\partial n = \mathbf{n} \cdot \nabla$. We shall define the Green's function, $G = G(\mathbf{x}, \mathbf{x}_0, t)$, for Dirichlet boundary condition on S . It is understood that $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ so that \mathbf{x}_0 is a fixed parameter of the problem. The independent variables are $(\mathbf{x}, t) = (x, y, t)$; the dependent variable is G .

LAPLACE EQUATION

$0 = \nabla^2 G + \delta(\mathbf{x} - \mathbf{x}_0)$ with $G = 0$ on S . Of course, G is independent of time.

DIFFUSION EQUATION

$G_t = \nabla^2 G + \delta(\mathbf{x} - \mathbf{x}_0)\delta(t)$ with $G = 0$ on S . The initial condition at $t = 0^-$ is $G = 0$. It is possible to *shift* the initial condition to $t = 0^+$ by integrating the PDE across $t = 0$; the result is $G(\mathbf{x}, \mathbf{x}_0, 0^+) = \delta(\mathbf{x} - \mathbf{x}_0)$.

WAVE EQUATION

$G_{tt} = \nabla^2 G + \delta(\mathbf{x} - \mathbf{x}_0)\delta(t)$ with $G = 0$ on S . The initial condition at $t = 0^-$ is $G = G_t = 0$. It is possible to *shift* the initial condition to $t = 0^+$ by integrating the PDE across $t = 0$; the result is $G(\mathbf{x}, \mathbf{x}_0, 0^+) = 0$ and $G_t(\mathbf{x}, \mathbf{x}_0, 0^+) = \delta(\mathbf{x} - \mathbf{x}_0)$. The PDE is $G_{tt} = \nabla^2 G$ for $t > 0$.

It is possible to represent the solutions of these equations (Laplace, diffusion and wave) with arbitrary boundary and initial conditions in terms of their Green's functions. It is simpler to obtain the latter because they satisfy null auxiliary conditions and have a simple impulsive forcing term.

LAPLACE EQUATION – Representation of the Solution by Green's Function

Consider the well posed Dirichlet problem $0 = \nabla^2 u + h$ with $u = g(\mathbf{x})$ on S . Here $h = h(\mathbf{x})$ is a given forcing term. We wish to obtain the representation of $u = u(\mathbf{x})$ in terms of the Green's function.

We take the inner product of the PDE with the Green's function and use Green's identity for the first term on the right-hand side. The result is

$$0 = \int_S \left(\bar{G} \frac{\partial u}{\partial n} - u \frac{\partial \bar{G}}{\partial n} \right) dS + u \cdot \nabla^2 G + h \cdot G$$

The equation above simplifies to

$$u(\mathbf{x}_0) = - \int_S \left(g(\mathbf{x}) \frac{\partial \bar{G}(\mathbf{x}, \mathbf{x}_0)}{\partial n_x} \right) dS_x + h \cdot G$$

where the subscript x is a reminder that the normal derivative and the surface integral are to be evaluated with respect to \mathbf{x} (and not \mathbf{x}_0). Of course, $h \cdot G = \int_{\mathcal{V}} h(\mathbf{x}) \bar{G}(\mathbf{x}, \mathbf{x}_0) d\mathcal{V}_x$ and G is a real function so that complex conjugation plays no role here.

We recall from specific examples that G is symmetric with respect to the field, \mathbf{x} , and source, \mathbf{x}_0 , points, $G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x})$, so that it is actually possible to carry out the integration with respect to the second argument of G in view of this symmetry owing to the self-adjointness of the Laplacian under the standard inner product. We shall not do the simple details (but the reader should).

Also observe that for a given well-posed problem for u , Green's identity provides null boundary condition that must be satisfied by G so that the problem for u becomes directly solvable.

WAVE EQUATION – Representation of the Solution by Green's Function

Consider the well posed Dirichlet problem $u_{tt} = \nabla^2 u + h$ with $u = g(\mathbf{x}, t)$ on S . Here $h = h(\mathbf{x}, t)$ is a given forcing term. The given initial conditions at $t = 0$ are $u = \phi(\mathbf{x})$ and $u_t = \psi(\mathbf{x})$. We wish to obtain the representation of $u = u(\mathbf{x}, t)$ in terms of the Green's function.

We take the inner product of the PDE with the Green's function, $G(\mathbf{x}, \mathbf{x}_0, t - \tau)$ and use Green's identity for the first term on the right-hand side. There is one caveat. The time variable in the PDE for u is τ . This is because we need a dummy variable (say τ) in order to integrate the forcing term to the current time, t . The first intermediate result is

$$u_{\tau\tau} \cdot G = \int_S \left(\bar{G} \frac{\partial u}{\partial n} - u \frac{\partial \bar{G}}{\partial n} \right) dS + u \cdot \nabla^2 G + h \cdot G$$

The equation above reduces to the governing equation

$$\frac{\partial}{\partial \tau} (u_\tau \bullet G) - u_\tau \bullet G_\tau = - \int_S \left(g(\mathbf{x}, \tau) \frac{\partial \bar{G}(\mathbf{x}, \mathbf{x}_0, t - \tau)}{\partial n_x} \right) dS_x + \frac{\partial}{\partial \tau} (u \bullet G_\tau) - u_\tau \bullet G_\tau - u(\mathbf{x}_0, \tau) \delta(t - \tau) + h \bullet G$$

in view of the facts that $G_{tt} = G_{\tau\tau}$ and $u \bullet \delta(\mathbf{x} - \mathbf{x}_0) = u(\mathbf{x}_0, \tau)$. It is understood that $t > 0$.

We now integrate the governing equation with respect to τ from 0 to $t^- = t - \varepsilon$, $\varepsilon \rightarrow 0^+$. The tricky terms are those of the τ -derivatives because of the behavior of G in time. The integral of the first term on the left-hand side is $-\int_{\mathcal{V}} u_\tau(\mathbf{x}, 0) \bar{G}(\mathbf{x}, \mathbf{x}_0, t) d\mathcal{V}_x = -\psi \bullet G(t)$ because $G = 0$ at the upper limit where the value of the time argument is $t - (t - \varepsilon) = \varepsilon \rightarrow 0^+$. Note the meaning of the abbreviated notation, $\psi \bullet G(t)$.

The τ -integral of the second term on the right-hand side is more subtle. We have the contributions at the upper and lower limits, respectively, as

$$\int_{\mathcal{V}} u(\mathbf{x}, t^-) \bar{G}_\tau(\mathbf{x}, \mathbf{x}_0, \varepsilon \rightarrow 0^+) d\mathcal{V}_x - \int_{\mathcal{V}} u(\mathbf{x}, 0) \bar{G}_\tau(\mathbf{x}, \mathbf{x}_0, t) d\mathcal{V}_x = -u(\mathbf{x}_0, t^-) + \phi \bullet G_t(t)$$

because $G_\tau(\mathbf{x}, \mathbf{x}_0, 0^+) = -G_t(\mathbf{x}, \mathbf{x}_0, 0^+) = -\delta(\mathbf{x} - \mathbf{x}_0)$; more generally $G_\tau = -G_t$. The integral of the fourth term on the right-hand side is zero.

Putting these partial results together, we find

$$u(\mathbf{x}_0, t^-) = \phi \bullet G_t(t) + \psi \bullet G(t) - \int_0^{t^-} \int_S \left(g(\mathbf{x}, \tau) \frac{\partial \bar{G}(\mathbf{x}, \mathbf{x}_0, t - \tau)}{\partial n_x} \right) dS_x d\tau + \int_0^{t^-} h(\tau) \bullet G(t - \tau) d\tau$$

There is no difficulty in replacing t by t^- in any of the G 's for positive values of t because G and its derivatives are continuous. In other words, it follows after renaming the time variable (i.e., $t^- \rightarrow t$)

$$u(\mathbf{x}_0, t) = \phi \bullet G_t(t) + \psi \bullet G(t) - \int_0^t \int_S \left(g(\mathbf{x}, \tau) \frac{\partial \bar{G}(\mathbf{x}, \mathbf{x}_0, t - \tau)}{\partial n_x} \right) dS_x d\tau + \int_0^t h(\tau) \bullet G(t - \tau) d\tau$$

This is the representation of the solution at any point (\mathbf{x}_0, t) in terms of the Green's function. Of course, the meaning of the last integral is $\int_0^t \int_{\mathcal{V}} h(\mathbf{x}, \tau) \bar{G}(\mathbf{x}, \mathbf{x}_0, t - \tau) d\mathcal{V}_x d\tau$. It is an inner product in space and a *convolution* in time.

It is also possible to obtain the final result by integrating with respect to τ from 0 to $t^+ = t + \varepsilon$, $\varepsilon \rightarrow 0^+$. Some of the terms will contribute differently. For example, the contribution of the fourth term on the right-hand side of the governing equation is

$-u(\mathbf{x}_0, t^+)$ (instead of zero, as before). The reader should carry out the details for all terms and finish the analysis.

DIFFUSION EQUATION – Representation of the Solution by Green’s Function

Consider the well posed Dirichlet problem $u_t = \nabla^2 u + h$ with $u = g(\mathbf{x}, t)$ on S . Here $h = h(\mathbf{x}, t)$ is a given forcing term. The given initial condition at $t = 0$ is $u = \phi(\mathbf{x})$. We wish to obtain the representation of $u = u(\mathbf{x}, t)$ in terms of the Green’s function.

The procedure is the same as that for the wave equation – it is a bit simpler because the PDE has only a first time derivative. We omit details which should be filled in by the reader. The final result is

$$u(\mathbf{x}_0, t) = \phi \bullet G(t) - \int_0^t \int_S \left(g(\mathbf{x}, \tau) \frac{\partial \bar{G}(\mathbf{x}, \mathbf{x}_0, t - \tau)}{\partial n_x} \right) dS_x d\tau + \int_0^t h(\tau) \bullet G(t - \tau) d\tau$$

Of course, the meaning of the last integral is $\int_0^t \int_{\mathcal{V}} h(\mathbf{x}, \tau) \bar{G}(\mathbf{x}, \mathbf{x}_0, t - \tau) d\mathcal{V}_x d\tau$.

Another good set of exercises is the derivation of the representation of the solutions of these equations in terms of the Green’s functions for the Neumann boundary condition.

Formula for the Green’s Function

In AME 500A we obtained Green’s functions for two point boundary value problems for ODE’s by eigenfunction expansion and by directly solving the ODE’s. In AME 500B we obtained the mini-Green for initial value problems for ODE’s and the Green’s functions for PDE’s by selecting $h(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}_0)\delta(t)$ in the general representation of the solution of the PDE’s.

We now illustrate a related method in which we *compare* two solutions of the PDE from which the Green’s function may be read off. We illustrate with the wave equation. The solution of the wave equation in terms of the Green’s function is

$$u(\mathbf{x}_0, t) = \phi \bullet G_t(t) + \psi \bullet G(t) - \int_0^t \int_S \left(g(\mathbf{x}, \tau) \frac{\partial \bar{G}(\mathbf{x}, \mathbf{x}_0, t - \tau)}{\partial n_x} \right) dS_x d\tau + \int_0^t h(\tau) \bullet G(t - \tau) d\tau$$

On the other hand the solution of the same equation by another method (say eigenfunction expansion) is

$$u(\mathbf{x}_0, t) = \sum_n u_n(\mathbf{x}_0) \left\{ (\phi \bullet u_n) \dot{G}(t) + (\psi \bullet u_n) G(t) + \int_0^t (h \bullet u_n)(\tau) G(t-\tau) d\tau \right\} \\ - \sum_n u_n(\mathbf{x}_0) \left\{ \int_S \frac{\partial \bar{u}_n(\mathbf{x})}{\partial n_x} \int_0^t g(\mathbf{x}, \tau) G(t-\tau) d\tau dS_x \right\}$$

where $G(t) = \sin(\sqrt{\lambda_n} t) / \sqrt{\lambda_n}$ is the mini-Green.

These two representations for $u(\mathbf{x}_0, t)$ are valid identically for all ϕ, ψ, g and h .

Therefore, $\psi \bullet G(t) = \sum_n u_n(\mathbf{x}_0) (\psi \bullet u_n) G(t)$. It follows that

$$\int_{\mathcal{V}} \psi(\mathbf{x}, t) \bar{G}(\mathbf{x}, \mathbf{x}_0, t) d\mathcal{V}_x = \int_{\mathcal{V}} \psi(\mathbf{x}, t) d\mathcal{V}_x \sum_n u_n(\mathbf{x}_0) \bar{u}_n(\mathbf{x}) \sin(\sqrt{\lambda_n} t) / \sqrt{\lambda_n}$$

and that $G(\mathbf{x}, \mathbf{x}_0, t) = \sum_n \bar{u}_n(\mathbf{x}_0) u_n(\mathbf{x}) \sin(\sqrt{\lambda_n} t) / \sqrt{\lambda_n}$, $t > 0$.

As a reminder we note that $u_n(\mathbf{x})$ are the orthonormal eigenfunctions of the (negative) Laplacian operator, $L = -(\partial^2 / \partial x^2 + \partial^2 / \partial y^2) = -\nabla \cdot \nabla$ (i.e., divergence of a gradient); $L(u_n) = \lambda_n u_n$ and $u_n = 0$ on S . L is self-adjoint under the standard inner product $f \bullet g = \int_{\mathcal{V}} f(\mathbf{x}) \bar{g}(\mathbf{x}) d\mathcal{V}$. Hence we have an orthonormal basis of eigenfunctions $B_{\perp} = \{u_n\}$; the corresponding (real) eigenvalues are λ_n , $n = 1, 2, \dots$.

Using this orthonormal basis we find, either for null Dirichlet or Neumann boundary conditions, that $L(u_n) \bullet u_n = \lambda_n$ or $\int_{\mathcal{V}} (\nabla u_n) \cdot (\nabla \bar{u}_n) d\mathcal{V} = \lambda_n$ so that λ_n is nonnegative; $\lambda_0 = 0$

must correspond to the eigenfunction $u_0 = \text{const} \neq 0$ associated with the Neumann problem. Hence even if u_n were complex, both the real and imaginary parts would have to be eigenfunctions corresponding to the eigenvalue λ_n . Using the Gramm-Schmidt orthogonalization procedure and the inner product $f \bullet g = \int_{\mathcal{V}} f(\mathbf{x}) \bar{g}(\mathbf{x}) d\mathcal{V} = \int_{\mathcal{V}} f(\mathbf{x}) g(\mathbf{x}) d\mathcal{V}$

for the (real) eigenfunctions, we can construct an orthonormal basis consisting of real eigenfunctions. Because of this, the symmetry (not the conjugate symmetry) of G follows for an interchange of \mathbf{x} and \mathbf{x}_0 .