2/7/08

HEAT CONDUCTION EQUATION 2D – Dirichlet boundary condition

Solution of the diffusion equation (parabolic PDE) with BC and IC on the domain $\mathcal{V}$ with boundary S; the outward unit normal is $\mathbf{n}$. We use the notation $\partial/\partial n = \mathbf{n}\cdot\nabla$.

A well posed problem is $u_t = u_{xx} + u_{yy} + h$ for $u = u(x, y, t)$. The given IC is: at $t = 0, u = \phi(x, y)$. The given BC is: on S: $u = g_1(x, y, t)$. The forcing term, $h = h(x, y, t)$, is also given.

We will use the short-hand notation $(x, y) = \mathbf{x}$ for the Cartesian coordinates x and y so that $u(x, y, t) = u(\mathbf{x}, t)$, etc.

The solution $(t > 0)$ is given by

$$u(\mathbf{x},t) = \sum_n u_n(\mathbf{x}) \left\{ (\phi \bullet u_n)\,\mathsf{G}(t) + \int_0^t (h \bullet u_n)(\tau)\,\mathsf{G}(t-\tau)\,d\tau - \int_S \frac{\partial \overline{u}_n(\mathbf{x}')}{\partial n} \int_0^t g_1(\mathbf{x}',\tau)\,\mathsf{G}(t-\tau)\,d\tau\, dS' \right\}$$

where $\mathsf{G}(t) = \exp(-\lambda_n t)$.

The (negative) Laplacian operator, $L = -(\partial^2/\partial x^2 + \partial^2/\partial y^2) = -\nabla\cdot\nabla$ (i.e., divergence of a gradient), is self-adjoint under the standard inner product $\int_{\mathcal{V}} f(\mathbf{x})\overline{g}(\mathbf{x})\,d\mathcal{V}$. Hence we have an orthonormal basis of eigenfunctions $B_\perp = \{u_n\}$; the corresponding (real) eigenvalues are $\lambda_n, n = 1, 2, \ldots$. Of course, $u_n = u_n(\mathbf{x})$, $L(u_n) = \lambda_n u_n$ and $u_n = 0$ on S.

Example: Suppose the domain $\mathcal{V}$ is the rectangle defined by the inequalities $0 < x < a$ and $0 < y < b$. The orthonormal eigenfunctions, u_n, can be constructed by multiplying together the corresponding "one-dimensional" eigenfunctions (essentially separation of variables). It is convenient (i.e., simpler) to label the orthonormal eigenfunctions and eigenvalues using double subscripts because of the "Cartesian product" nature of the domain. The result is

$$u_{pq} = (2/\sqrt{ab})\sin(p\pi x/a)\sin(q\pi y/b), \quad \lambda_{pq} = (p\pi/a)^2 + (q\pi/b)^2 \;, \quad p, q = 1, 2, 3...$$

We can actually arrange the double sequence u_{pq} as a single sequence u_n by the ingenious trick invented by Cantor. But we do not have to do this; in the equation above for $u(\mathbf{x}, t)$, $\sum_n$ implies summation over **all** eigenfunctions. This translates into the double sum $\sum_p \sum_q \; = \sum_q \sum_p$. If all else fails, this sum can be added up on the computer to include a finite number of terms as an approximation of the solution.