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## LAPLACE EQUATION 2D – Dirichlet boundary condition

**1.** Solution of the Laplace (Poisson) equation (elliptic PDE) with BC on the domain  $\mathcal{V}$  with boundary  $S$ ; the outward unit normal is  $\mathbf{n}$ . We use the notation  $\partial/\partial n = \mathbf{n} \cdot \nabla$ .

A well posed problem is  $0 = u_{xx} + u_{yy} + h$  for  $u = u(x, y)$ . The given BC is: on  $S$ :  $u = g_1(x, y)$ . The forcing term,  $h = h(x, y)$ , is also given.

We will use the short-hand notation  $(x, y) = \mathbf{x}$  for the Cartesian coordinates  $x$  and  $y$  so that  $u(x, y) = u(\mathbf{x})$ , etc.

The solution is given by

$$u(\mathbf{x}) = \sum_n u_n(\mathbf{x}) \left\{ \lambda_n^{-1} (h \cdot u_n) - \lambda_n^{-1} \int_S \frac{\partial \bar{u}_n(\mathbf{x}')}{\partial n} g_1(\mathbf{x}') dS' \right\}$$

provided that  $\lambda_n \neq 0$  (which is indeed the case). More about this shortly and also in connection with the Helmholtz equation.

The (negative) Laplacian operator,  $L = -(\partial^2/\partial x^2 + \partial^2/\partial y^2) = -\nabla \cdot \nabla$  (i.e., divergence of a gradient), is self-adjoint under the standard inner product  $f \cdot g = \int_{\mathcal{V}} f(\mathbf{x}) \bar{g}(\mathbf{x}) d\mathcal{V}$ . Hence we have an orthonormal basis of eigenfunctions  $B_{\perp} = \{u_n\}$ ; the corresponding (real) eigenvalues are  $\lambda_n, n = 1, 2, \dots$ . Of course,  $u_n = u_n(\mathbf{x})$ ,  $L(u_n) = \lambda_n u_n$  and  $u_n = 0$  on  $S$ .

**Example:** Suppose the domain  $\mathcal{V}$  is the rectangle defined by the inequalities  $0 < x < a$  and  $0 < y < b$ . The orthonormal eigenfunctions,  $u_n$ , can be constructed by multiplying together the corresponding “one-dimensional” eigenfunctions (essentially separation of variables). It is convenient (i.e., simpler) to label the orthonormal eigenfunctions and eigenvalues using double subscripts because of the “Cartesian product” nature of the domain. The result is

$$u_{pq} = (2/\sqrt{ab}) \sin(p\pi x/a) \sin(q\pi y/b), \quad \lambda_{pq} = (p\pi/a)^2 + (q\pi/b)^2 \neq 0, \quad p, q = 1, 2, 3, \dots$$

We can actually arrange the double sequence  $u_{pq}$  as a single sequence  $u_n$  by the ingenious trick invented by Cantor. But we do not have to do this; in the equation above for  $u(\mathbf{x})$ ,  $\sum_n$  implies summation over **all** eigenfunctions. This translates into the double sum  $\sum_p \sum_q = \sum_q \sum_p$ . If all else fails, this sum can be added up on the computer to include a finite number of terms as an approximation of the solution.

## LAPLACE EQUATION 2D – Neumann boundary condition

2. Solution of the Laplace (Poisson) equation (elliptic PDE) with BC on the domain  $\mathcal{V}$  with boundary  $S$ ; the outward unit normal is  $\mathbf{n}$ . We use the notation  $\partial / \partial n = \mathbf{n} \cdot \nabla$ .

A well posed problem is  $0 = u_{xx} + u_{yy} + h$  for  $u = u(x, y)$ . The given BC is: on  $S$ :  $\partial u / \partial n = g_2(x, y)$ . The forcing term,  $h = h(x, y)$ , is also given.

We will use the short-hand notation  $(x, y) = \mathbf{x}$  for the Cartesian coordinates  $x$  and  $y$  so that  $u(x, y) = u(\mathbf{x})$ , etc.

At this time we do have to face the existence of zero eigenvalues. Because of this, a solvability issue arises. For those values of  $n$  for which  $\lambda_n = 0$  we **must** have

$$(h \bullet u_n) + \int_S \bar{u}_n(\mathbf{x}') g_2(\mathbf{x}') dS' = 0$$

Otherwise the problem has **no** solution. If the *necessary* condition above is satisfied,  $u \bullet u_n$  cannot be determined for those  $n$  for which  $\lambda_n = 0$ . In other words the solution  $u(\mathbf{x})$  exists, but has some arbitrariness.

This solution is given by

$$u(\mathbf{x}) = \sum_n u_n(\mathbf{x}) \left\{ \lambda_n^{-1} (h \bullet u_n) + \lambda_n^{-1} \int_S \bar{u}_n(\mathbf{x}') g_2(\mathbf{x}') dS' \right\} + \dots$$

where the sum is taken over those  $n$  for which  $\lambda_n \neq 0$ . The ellipses stand for a sum of the form

$$\sum_{k, \lambda_k=0} \alpha_k u_k(\mathbf{x})$$

where the  $\alpha$ 's are arbitrary.

The (negative) Laplacian operator,  $L = -(\partial^2 / \partial x^2 + \partial^2 / \partial y^2) = -\nabla \cdot \nabla$  (i.e., divergence of a gradient), is self-adjoint under the standard inner product  $f \bullet g = \int_{\mathcal{V}} f(\mathbf{x}) \bar{g}(\mathbf{x}) d\mathcal{V}$ . Hence we have an orthonormal basis of eigenfunctions  $B_{\perp} = \{u_n\}$ ; the corresponding (real) eigenvalues are  $\lambda_n, n = 1, 2, \dots$ . Of course,  $u_n = u_n(\mathbf{x})$ ,  $L(u_n) = \lambda_n u_n$  and  $\partial u_n / \partial n = 0$  on  $S$ .

**Example:** Suppose the domain  $\mathcal{V}$  is the rectangle defined by the inequalities  $0 < x < a$  and  $0 < y < b$ . The orthonormal eigenfunctions,  $u_n$ , can be constructed by multiplying together the corresponding “one-dimensional” eigenfunctions (essentially separation of variables). It is convenient (i.e., simpler) to label the orthonormal eigenfunctions and eigenvalues using double subscripts because of the “Cartesian product” nature of the domain. The result is

$$u_{pq} = (2/\sqrt{ab}) \varepsilon_p \varepsilon_q \cos(p\pi x/a) \cos(q\pi y/b), \quad \lambda_{pq} = (p\pi/a)^2 + (q\pi/b)^2, \quad p, q = 0, 1, 2, \dots$$

where  $\varepsilon_0 = 1/\sqrt{2}$  and  $\varepsilon_k = 1, k \geq 1$ . This  $\varepsilon$ -symbol enables us to write all eigenfunctions as “cosines” and to guarantee that each eigenvector has unit norm (e.g.,  $u_{00} = 1/\sqrt{ab}$ ).

We have only one zero eigenvalue, namely,  $\lambda_{00}$ . Therefore, the arbitrariness in the Neumann problem (at least for a rectangle) is a constant. Actually, the last statement is true for any domain.

As an aside, we note that the integration of the given PDE over the domain implies the necessary condition for the existence of a solution

$$0 = \int_S g_2(\mathbf{x}) dS + \int_{\mathcal{V}} h(\mathbf{x}) d\mathcal{V}$$

The equation above is identical to the previously stated necessary condition implied by the special eigenfunction (call it  $u_1$ ),  $u_1 = const$ , that always corresponds to the eigenvalue,  $\lambda_1 = 0$ . More mathematical books on the subject show that this is the **only** eigenfunction corresponding to this eigenvalue.

In a physical problem, the necessary condition in integral form is interpreted as a “conservation” equation.

The moral of the story is that a linear operator that has a zero eigenvalue is not “invertible”; solvability issues always arise in the corresponding inhomogeneous equation and the solution is never unique (if there is one at all) – AME 500A.

From a physical consideration, the arbitrariness in the solution of the Neumann problem is not a serious issue. Often one is interested in  $\nabla u$ , rather than  $u$  itself. Even in problems in which  $u$  itself is of some significance, the “zero potential” can be assigned arbitrarily because it is *differences* in potential that are important (e.g., difference in gravitational potential).