

Solution of ODE by Laplace Transform – Double Pole

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We now continue with the solution of the ODE for the case when one of the characteristic roots of the coefficient matrix has multiplicity two and this matrix cannot be diagonalized.

ODE

The standard form of a linear ODE is $dy/dt = Ay + f(t)$ for the dependent variable, $y(t) = (y_1, y_2, \dots, y_n)$; the independent variable is t (time-like). The $(n \times n)$ matrix A is given with constant elements and the forcing term, $f(t)$, is an arbitrary (given) function of time for which the indicated operations below are meaningful. A well posed problem includes the initial condition at $t = 0$ specified by $y(0) = y_0 = \text{given}$. We are interested in the solution of the ODE for $t > 0$, satisfying this given initial condition.

Solution of ODE by Laplace Transform

The one-sided Laplace transform of $f = f(t)$ is defined by $\hat{f}(s) = \int_0^\infty f(t)e^{-st} dt$ where it is understood that $f = 0$ for $t < 0$. To recover $f(t)$ from $\hat{f}(s)$ we use the Bromwich contour in the complex s -plane; specifically, $f(t) = \frac{1}{2\pi i} \int_{Br} \hat{f}(s)e^{st} ds$ where

$\int_{Br} = \int_{\gamma-i\infty}^{\gamma+i\infty}$ for γ sufficiently positive. In other words, $\hat{f}(s)$ is analytic in the right-half of the s -plane as separated by Br which is a line parallel to the imaginary axis. This ensures that $f(t) = 0$ for $t < 0$ as seen by closing the contour along a large semicircle to the right of Br . The Laplace transform is essentially a Fourier transform as explained in class.

An application of the Laplace transform to the ODE yields $s\hat{y} - y(0) = A\hat{y} + \hat{f}$ where the left-hand side of this equation comes from integration by parts. In the process we pick up the initial condition, $y(0) = y_0$.

The first equation in the preceding paragraph is an algebraic equation for the transform, \hat{y} . Its solution is $-\hat{y} = (A - sI)^{-1}y_0 + (A - sI)^{-1}\hat{f}$. It follows from the inverse Laplace transform that the solution is

$$-y(t) = \frac{1}{2\pi i} \int_{Br} e^{st} (A - sI)^{-1} y_0 ds + \frac{1}{2\pi i} \int_{Br} e^{st} (A - sI)^{-1} \int_0^\infty e^{-s\tau} f(\tau) d\tau ds, \quad t > 0$$

We are now ready to evaluate these s -integrals by the method of contour integration. In order to do this we must find the poles (the “infinities”) of the integrands. We note that $(A - sI)^{-1} = (A - sI)_{\text{mod}} / \det(A - sI)$ where $(A - sI)_{\text{mod}}$ is finite. Thus, the poles of the integrand will occur at the zeroes of $\det(A - sI)$. But, $\det(A - sI) = (-1)^n (s^n + \dots) = (-1)^n (s - s_1)(s - s_2) \dots (s - s_n)$ is a polynomial of degree n whose n roots are $\{s_1, s_2, \dots, s_n\}$.

Here we shall treat the case where $\det(A - sI) = (-1)^n (s - s_1)^2 (s - s_3) \dots (s - s_n)$ so that $s = s_1$ is a double root (i.e., $s_1 = s_2$); all other roots are of multiplicity one.

The integral $\int_0^\infty \dots d\tau$ requires a careful look. Clearly, $\int_0^\infty = \int_0^t + \int_t^\infty$. In the third integral $\tau > t$ so that the s -integral along Br must be closed along a large semicircle to the right of Br [where the exponent of $e^{s(t-\tau)}$ is negative]; the result is that the third integral vanishes or that $\int_0^\infty \dots d\tau = \int_0^t \dots d\tau$ because all the singularities of the integrand are to the left of Br . In other words, the solution at current time, t , may be influenced by the past but not by the future (at least this is what we believe). This is the principle of *causality*. For all other integrals we close the Bromwich contour by a large semicircle on the left; the contribution to the integral along this semicircle is zero.

The solution by the method of residues at simple poles (as in the previous handout) is

$$-\mathbf{y}(t) = \sum_{k=3}^n e^{s_k t} \lim_{s \rightarrow s_k} [(s - s_k)(A - sI)^{-1}] \mathbf{y}_0 + \sum_{k=3}^n \lim_{s \rightarrow s_k} [(s - s_k)(A - sI)^{-1}] \int_0^t e^{s_k(t-\tau)} \mathbf{f}(\tau) d\tau + \dots$$

where the ellipsis stand for the contribution from the double pole at $s = s_1$.

To obtain the latter, we write $(A - sI)^{-1} = (A - sI)_{\text{mod}} / \det(A - sI)$ or

$$(A - sI)^{-1} = (A - sI)_{\text{mod}} / [(-1)^n (s - s_1)^2 (s - s_3) \dots (s - s_n)] = B(s) / (s - s_1)^2$$

where $B(s) = (A - sI)_{\text{mod}} / [(-1)^n (s - s_3) \dots (s - s_n)]$ is analytic at $s = s_1$. The residue of $B(s)e^{st} / (s - s_1)^2$ at $s = s_1$ is readily read off from the Taylor series of the numerator

$$\{B(s_1) + B'(s_1)(s - s_1) + \dots\} \{e^{s_1 t} + t e^{s_1 t} (s - s_1) + \dots\} / (s - s_1)^2$$

as $[B'(s_1)e^{s_1 t} + B(s_1)t e^{s_1 t}] = \partial / \partial s [B(s)e^{st}]_{s=s_1} = \lim_{s \rightarrow s_1} \partial / \partial s [(s - s_1)^2 (A - sI)^{-1} e^{st}]$.

We are done. There are two observations. First, the double root (or the double pole) spawns the two linearly independent solutions $B'(s_1)e^{s_1 t}$ and $B(s_1)t e^{s_1 t}$. The latter is called *secular*. If we had a triple pole at $s = s_1$, the three linearly independent solutions corresponding to this root are $\exp(s_1 t)$, $t \exp(s_1 t)$ and $t^2 \exp(s_1 t)$. We are unable to get these secular solutions from the eigenmethod solution of the ODE because we have not done (in AME 500A) an in-depth study of the Jordan form of A . Second, the solution for $-\mathbf{y}(t)$ is the boxed expression above to which we **add** the contribution from the double pole that reads as

$$\dots + \lim_{s \rightarrow s_1} \partial / \partial s [(s - s_1)^2 (A - sI)^{-1} e^{st}] \mathbf{y}_0 + \int_0^t \lim_{s \rightarrow s_1} \partial / \partial s [(s - s_1)^2 (A - sI)^{-1} e^{s(t-\tau)}] \mathbf{f}(\tau) d\tau$$

Hopefully the reader sees how to treat the case when distinct roots of $\det(A - sI)$, $\{s_1^*, s_2^*, \dots, s_d^*\}$, have arbitrary multiplicities. In this case, $\det(A - sI) = (-1)^n (s - s_1^*)^{m_1} (s - s_2^*)^{m_2} \dots (s - s_d^*)^{m_d}$ where $\{m_1 \geq 1, m_2 \geq 1, \dots, m_d \geq 1\}$ are the multiplicities (or the order of the respective poles). Of course, $m_1 + m_2 + \dots + m_d = n$. On p. 278 Greenberg, an expression for the residue at a pole of any order is given in equation 15.8.