1/31/08

WAVE EQUATION 1D – Dirichlet boundary condition

Solution of the wave equation (hyperbolic PDE) with BC and IC on the domain $0 < x < l$.

A well posed problem is $u_{tt} = u_{xx} + h$ for $u = u(x,t)$. The given IC is: at $t = 0, u = \phi(x)$ and $u_t = \psi(x)$. The given BC are: at $x = 0,\ u = g_1(t)$ and at $x = l,\ u = g_2(t)$. The forcing term, $h = h(x,t)$, is also given.

The solution $(t > 0)$ is given by

$$u(x,t) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} \left\{ (\phi \bullet u_n)\dot{\mathsf{G}}(t) + (\psi \bullet u_n)\mathsf{G}(t) - \overline{u}_n'(l) \int_0^t g_2(\tau)\mathsf{G}(t-\tau)d\tau \right\}$$

$$+ \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} \left\{ \overline{u}_n'(0) \int_0^t g_1(\tau)\mathsf{G}(t-\tau)d\tau + \int_0^t (h \bullet u_n)(\tau)\mathsf{G}(t-\tau)d\tau \right\}$$

The orthonormal eigenfunctions for the problem are $u_n(x) = \sqrt{2/l}\sin(n\pi x/l)$; the corresponding eigenvalues are $\lambda_n = (n\pi/l)^2$, $n = 1,2,3,\ldots$. The standard inner product, $f \bullet g = \int_0^l f(x)\overline{g}(x)dx$ is understood.

$\mathsf{G}(t) = \sin(n\pi t/l)/(n\pi/l)$ is the "mini-Green's" function as discussed in class ($\dot{\mathsf{G}} = d\mathsf{G}/dt$) for the initial value problem for $u \bullet u_n$.

Example 1: $g_1 = g_2 = \phi = \psi = 0$ and $h = \delta(x - x_0)\delta(t - 0^+)$. This is the Green's function, $G(x, x_0, t)$, for the wave equation.

Example 2: $g_1(t) = \sin(2\pi t)H(2-t)$, $g_2 = \phi = \psi = h = 0$. One end of the string is moved up and down for two cycles.

Example 3: $g_1 = g_2 = \phi = h = 0$, $\psi(x) = H[0.01 - (x - x_0)^2]$. The string is given a constant $(=1)$ initial velocity in the interval $|x - x_0| < 0.1$.

NOTE: H denotes the Heaviside function.